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On uniqueness conditions for Candecomp/Parafac and Indscal with full column rank in one mode ☆

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ABSTRACT

In the Candecomp/Parafac (CP) model, a three-way array $\underline{\mathbf{X}}$ is written as the sum of R outer vector product arrays and a residual array. The former comprise the columns of the component matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. For fixed residuals, $(\mathbf{A},\mathbf{B},\mathbf{C})$ is unique up to trivial ambiguities, if $2R+2$ is less than or equal to the sum of the k-ranks of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. This classical result was shown by Kruskal in 1977. In this paper, we consider the case where one of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ has full column rank, and show that in this case Kruskal's uniqueness condition implies a recently obtained uniqueness condition. Moreover, we obtain Kruskal-type uniqueness conditions that are weaker than Kruskal's condition itself. Also, for $(\mathbf{A},\mathbf{B},\mathbf{C})$ with $\text{rank}(\mathbf{A})=R-1$ and $\mathbf{C}$ full column rank, we obtain easy-to-check necessary and sufficient uniqueness conditions. We extend our results to the Indscal decomposition in which the array $\underline{\mathbf{X}}$ has symmetric slices and $\mathbf{A}=\mathbf{B}$ is imposed. We consider the real-valued CP and Indscal decompositions, but our results are also valid for their complex-valued counterparts.

1. Introduction

Carroll and Chang [1] and Harshman [4] independently proposed the same method for component analysis of three-way arrays, and named it Candecomp and Parafac, respectively. Let $\circ$ denote the outer vector product, i.e., for vectors $\mathbf{x}$ and $\mathbf{y}$ we define $\mathbf{x}\circ\mathbf{y}=\mathbf{x}\mathbf{y}^T$. For three vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$, the product $\mathbf{x}\circ\mathbf{y}\circ\mathbf{z}$ is a three-way array with elements $x_iy_jz_k$. In the Candecomp/Parafac (CP) model, an $I\times J\times K$ array $\underline{\mathbf{X}}$ is decomposed into

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$$\underline{\mathbf{X}} = \sum_{r=1}^{R} (\mathbf{a}_r \circ \mathbf{b}_r \circ \mathbf{c}_r) + \underline{\mathbf{E}}, \tag{1.1}$$

where the vectors $\mathbf{a}_r$, $\mathbf{b}_r$ and $\mathbf{c}_r$ have size I, J and K, respectively. For fixed R, the CP decomposition (1.1) is found by minimizing the sum of squares of $\underline{\mathbf{E}}$. Usually an iterative algorithm is used for this purpose, see e.g. Tomasi and Bro [26]. In this paper, we will denote column vectors as $\mathbf{x}$, matrices as $\mathbf{X}$ and three-way arrays as $\underline{\mathbf{X}}$.

We consider the real-valued CP model, i.e. we assume the array $\underline{\mathbf{X}}$ and the vectors $\mathbf{a}_r$, $\mathbf{b}_r$ and $\mathbf{c}_r$ to be real-valued. The real-valued CP model is used in a majority of applications in psychology and chemistry; see Kroonenberg [10], Kiers and Van Mechelen [7] and Smilde et al. [16]. Complex-valued applications of CP occur in e.g. signal processing and telecommunications research; see Sidiropoulos et al. [14] and Sidiropoulos et al. [15]. For an overview of applications of the CP model and related models, see Kolda and Bader [8].

The CP model can be considered as a three-way generalization of Principal Component Analysis or the Singular Value Decomposition. The three-way rank of $\underline{\mathbf{X}}$ is defined as the smallest number of rank-1 arrays whose sum equals $\underline{\mathbf{X}}$. A three-way array has rank 1 if it is the outer product of three vectors. Hence, in the CP decomposition (1.1) each of the R components $(\mathbf{a}_r \circ \mathbf{b}_r \circ \mathbf{c}_r)$ has rank 1. The three-way rank of $\underline{\mathbf{X}}$ is equal to the smallest number of components for which a CP decomposition exists with perfect fit, i.e., with an all-zero residual term $\underline{\mathbf{E}}$. Moreover, a CP algorithm designed to minimize the sum of squares of the residuals tries to find a best rank-R approximation of the array $\underline{\mathbf{X}}$. Unfortunately, such a best rank-R approximation does not always exist, see De Silva and Lim [3]. In such cases, diverging CP components occur while running a CP algorithm. This phenomenon is also known as "degeneracy", see Kruskal et al. [12], Stegeman [17,19,18] and Krijnen et al. [9]. Since we consider the real-valued CP model, the rank of any array is assumed to be the rank over the real field.

A CP solution is usually expressed in terms of component matrices $\mathbf{A}(I \times R)$, $\mathbf{B}\,(J \times R)$ and $\mathbf{C}(K \times R)$, which have as columns the vectors $\mathbf{a}_r$, $\mathbf{b}_r$ and $\mathbf{c}_r$, respectively. Let the kth slices of $\underline{\mathbf{X}}$ and $\underline{\mathbf{E}}$ be denoted by $\mathbf{X}_k$ $(I \times J)$ and $\mathbf{E}_k$ $(I \times J)$, respectively. Then (1.1) can be written as

$$\mathbf{X}_k = \mathbf{A}\mathbf{C}_k\mathbf{B}^T + \mathbf{E}_k, \quad k = 1, \ldots, K, \tag{1.2}$$

where $\mathbf{C}_k$ is the diagonal matrix with the kth row of $\mathbf{C}$ as its diagonal.

One of the most attractive features of the CP model is its uniqueness property. The uniqueness of a CP solution is usually studied for given residuals $\underline{\mathbf{E}}$, or, equivalently, for a given fitted model array. To any set of component matrices $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ corresponds a fitted model array $\widehat{\underline{\mathbf{X}}} = \underline{\mathbf{X}} - \underline{\mathbf{E}}$. It can be seen that the component matrices $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ corresponding to $\widehat{\underline{\mathbf{X}}}$ can only be unique up to rescaling/counterscaling and jointly permuting columns of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. Indeed, the fitted model array will be the same for the solution given by $\overline{\mathbf{A}} = \mathbf{A}\mathbf{\Pi}\mathbf{T}_a$, $\overline{\mathbf{B}} = \mathbf{B}\mathbf{\Pi}\mathbf{T}_b$ and $\overline{\mathbf{C}} = \mathbf{C}\mathbf{\Pi}\mathbf{T}_c$, for a permutation matrix $\mathbf{\Pi}$ and diagonal matrices $\mathbf{T}_a, \mathbf{T}_b$ and $\mathbf{T}_c$ with $\mathbf{T}_a\mathbf{T}_b\mathbf{T}_c = \mathbf{I}_R$. When, for a given fitted model array $\widehat{\underline{\mathbf{X}}}$, the CP solution $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is unique up to these indeterminacies, it is called *essentially unique*.

Kruskal [11] has shown that essential uniqueness of the CP solution holds under relatively mild conditions. Kruskal's condition relies on a particular concept of matrix rank that he introduced, which has been named k-rank after him. Specifically, the k-rank of a matrix is the largest number x such that every subset of x columns of the matrix is linearly independent. We denote the k-rank of a matrix $\mathbf{A}$ as $k_{\mathbf{A}}$. For a CP solution $(\mathbf{A}, \mathbf{B}, \mathbf{C})$, Kruskal [11] proved that

$$k_{\mathbf{A}} + k_{\mathbf{B}} + k_{\mathbf{C}} \geqslant 2R + 2 \tag{1.3}$$

is a sufficient condition for essential uniqueness. A more condensed and accessible proof of (1.3) than Kruskal's original proof was given by Stegeman and Sidiropoulos [22]. Kruskal's uniqueness condition was generalized to N-way arrays with $N > 3$ by Sidiropoulos and Bro [13]. Ten Berge and Sidiropoulos [24] have shown that Kruskal's sufficient condition is also *necessary* for essential uniqueness when $R = 2$ or $R = 3$, but not when $R > 3$. It may be noted that (1.3) cannot be met when $R = 1$. However, uniqueness for that case has already been proven by Harshman [5]. Ten Berge and Sidiropoulos [24] conjectured that Kruskal's condition might be necessary and sufficient when $R > 3$, provided that the k-ranks of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are equal to their ranks. However, Stegeman and Ten Berge [20] refuted this conjecture.

Alternative CP uniqueness conditions were obtained by Jiang and Sidiropoulos [6] and De Lathauwer [2]. They independently examined the case where one of the component matrices, say $\mathbf{C}$, has full column rank. CP uniqueness then only depends on $(\mathbf{A}, \mathbf{B})$. Jiang and Sidiropoulos [6] obtained the following necessary and sufficient CP uniqueness condition:

$$(\mathbf{A} \odot \mathbf{B})\mathbf{d} \text{ is not of the form } (\mathbf{f} \otimes \mathbf{g}) \text{ for any } \mathbf{d} \text{ with } \omega(\mathbf{d}) \geqslant 2, \tag{1.4}$$

where $\mathbf{d} = (d_1, \ldots, d_R)^T$, $\omega(\cdot)$ denotes the number of nonzero elements of a vector, $\otimes$ denotes the Kronecker product, and $\odot$ denotes the (column-wise) Khatri-Rao product, i.e., $\mathbf{A} \odot \mathbf{B} = [\mathbf{a}_1 \otimes \mathbf{b}_1 | \cdots | \mathbf{a}_R \otimes \mathbf{b}_R]$.

Unfortunately, condition (1.4) is not easy to check. Jiang and Sidiropoulos [6] showed that (1.4) is equivalent to

$$\mathbf{U} \begin{pmatrix} d_1 d_2 \\ d_1 d_3 \\ \vdots \\ d_{R-1} d_R \end{pmatrix} = \mathbf{0} \text{ implies } \omega(\mathbf{d}) \leqslant 1, \tag{1.5}$$

where the matrix $\mathbf{U}$ depends on $(\mathbf{A}, \mathbf{B})$. This shows that $\mathbf{U}$ having full column rank is sufficient for condition (1.4) to hold. However, $\mathbf{U}$ having full column rank is not necessary for (1.4) to hold.

De Lathauwer [2] independently derived the same sufficient CP uniqueness condition, i.e. $\mathbf{U}$ having full column rank. Moreover, De Lathauwer [2] showed that if $(\mathbf{A}, \mathbf{B})$ is randomly sampled from an $(I+J)R$-dimensional continuous distribution, then $\mathbf{U}$ has full column rank almost surely if

$$\frac{I(I-1)J(J-1)}{4} \geqslant \frac{R(R-1)}{2}. \tag{1.6}$$

As will be explained in the next section, condition (1.6) is equivalent to $\mathbf{U}$ being a square or vertical matrix (after redundant rows have been deleted). An alternative proof of condition (1.6) was provided by Stegeman, Ten Berge and De Lathauwer [21]. Also, De Lathauwer [2] showed that if $\mathbf{C}$ has full column rank and the residual term is all-zero, then the CP decomposition of $\underline{\mathbf{X}}$ can be obtained algebraically from a simultaneous matrix diagonalization.

In this paper, we consider the relation between uniqueness condition (1.5) and

$$k_{\mathbf{A}} + k_{\mathbf{B}} \geqslant R + 2, \tag{1.7}$$

which is Kruskal's condition (1.3) when $k_{\mathbf{C}} = R$ ($\mathbf{C}$ has full column rank). For random $(\mathbf{A}, \mathbf{B})$ as above, condition (1.6) is more relaxed than Kruskal's condition (1.7), which equals $\min(I, R) + \min(J, R) \geqslant R + 2$ in this case. It is generally believed that this is also true for non-random $(\mathbf{A}, \mathbf{B})$, although a mathematical proof has not been provided. The main contribution of this paper is that we provide this mathematical proof. In particular, we show that (1.7) implies that $\mathbf{U}$ has full column rank. Accidently, we obtain several Kruskal-type uniqueness conditions that are more relaxed than (1.7) but stronger than $\mathbf{U}$ having full column rank. Also, for $(\mathbf{A}, \mathbf{B})$ with $\text{rank}(\mathbf{A}) = R - 1$, we obtain easy-to-check necessary and sufficient uniqueness conditions. Moreover, we extend our analysis to the Indscal model, which is the CP model for $I \times I \times K$ arrays with symmetric slices and the constraint $\mathbf{A} = \mathbf{B}$ imposed. Our results yield more insight into CP uniqueness and shed light on the space between Kruskal's condition (1.7) and the necessary and sufficient condition (1.5) of Jiang and Sidiropoulos [6].

The paper is organized as follows. In Section 2, we consider the structure of the matrix $\mathbf{U}$ of Jiang and Sidiropoulos [6] and present some auxiliary results. In Sections 3 and 4, we consider CP uniqueness when $\mathbf{C}$ has full column rank. In Section 5, we consider uniqueness of the Indscal decomposition. Finally, Section 6 contains a discussion of our results.

2. Structure of the matrix U

As mentioned above, the necesary and sufficient CP uniqueness condition (1.4) of Jiang and Sidiropoulos [6] holds if the matrix $\mathbf{U}$, depending on the elements of $\mathbf{A}$ and $\mathbf{B}$, has full column rank. Here, we consider the structure of $\mathbf{U}$. We have

$$\mathbf{U} = \widetilde{\mathbf{A}} \odot \widetilde{\mathbf{B}}, \tag{2.1}$$

where the matrices $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{B}}$ are defined as follows. Let $\widetilde{\mathbf{A}}$ have elements

$$\begin{vmatrix} a_{ig} & a_{ih} \\ a_{jg} & a_{jh} \end{vmatrix}, \quad \text{with } 1 \leqslant i < j \leqslant I \quad \text{and} \quad 1 \leqslant g < h \leqslant R, \tag{2.2}$$

where in each row of $\widetilde{\mathbf{A}}$ the value of (i,j) is fixed and in each column of $\widetilde{\mathbf{A}}$ the value of (g,h) is fixed. The columns of $\widetilde{\mathbf{A}}$ are ordered such that index g runs slower than h. The rows of $\widetilde{\mathbf{A}}$ are ordered such that index i runs slower than j. Note that $\widetilde{\mathbf{A}}$ can be written as

$$\widetilde{\mathbf{A}} = \mathbf{V}_1(\mathbf{A} \otimes \mathbf{A})\mathbf{W}, \tag{2.3}$$

where $\mathbf{V}_1$ contains $I(I-1)/2$ distinct rows of $\mathbf{I}_{I^2}$, and $\mathbf{W}$ contains $R(R-1)/2$ linearly independent columns of the form $\mathbf{e}_k - \mathbf{e}_l$, where the latter are columns of $\mathbf{I}_{R^2}$. The matrix $(\mathbf{A} \otimes \mathbf{A})\mathbf{W}$ contains the columns $(\mathbf{a}_g \otimes \mathbf{a}_h) - (\mathbf{a}_h \otimes \mathbf{a}_g)$ with $1 \leqslant g < h \leqslant R$. The premultiplication by $\mathbf{V}_1$ in (2.3) deletes rows from $(\mathbf{A} \otimes \mathbf{A})\mathbf{W}$ that are all-zero or identical (up to sign) to another row.

Analogous to $\widetilde{\mathbf{A}}$, let $\widetilde{\mathbf{B}}$ have elements

$$\begin{vmatrix} b_{ig} & b_{ih} \\ b_{jg} & b_{jh} \end{vmatrix}, \quad \text{with } 1 \leqslant i < j \leqslant J \quad \text{and} \quad 1 \leqslant g < h \leqslant R, \tag{2.4}$$

where in each row of $\widetilde{\mathbf{B}}$ the value of (i,j) is fixed and in each column of $\widetilde{\mathbf{B}}$ the value of (g,h) is fixed. The columns of $\widetilde{\mathbf{B}}$ are ordered such that index g runs slower than h. The rows of $\widetilde{\mathbf{B}}$ are ordered such that index i runs slower than j. As in (2.3), the matrix $\widetilde{\mathbf{B}}$ can be written as

$$\widetilde{\mathbf{B}} = \mathbf{V}_2(\mathbf{B} \otimes \mathbf{B})\mathbf{W}, \tag{2.5}$$

where $\mathbf{V}_2$ contains $J(J-1)/2$ distinct rows of $\mathbf{I}_{J^2}$.

The matrix $\mathbf{U}$ is defined by (2.1) up to a row permutation. From (2.3) and (2.5) we obtain

$$\mathbf{U} = (\mathbf{V}_1 \otimes \mathbf{V}_2)[((\mathbf{A} \otimes \mathbf{A})\mathbf{W}) \odot ((\mathbf{B} \otimes \mathbf{B})\mathbf{W})]. \tag{2.6}$$

The matrix $((\mathbf{A} \otimes \mathbf{A})\mathbf{W}) \odot ((\mathbf{B} \otimes \mathbf{B})\mathbf{W})$ contains the columns $[(\mathbf{a}_g \otimes \mathbf{a}_h) - (\mathbf{a}_h \otimes \mathbf{a}_g)] \otimes [(\mathbf{b}_g \otimes \mathbf{b}_h) - (\mathbf{b}_h \otimes \mathbf{b}_g)]$ with $1 \leqslant g < h \leqslant R$. The premultiplication by $(\mathbf{V}_1 \otimes \mathbf{V}_2)$ deletes rows which are all-zero or identical (up to sign) to another row. The matrix $\mathbf{U}$ in Jiang and Sidiropoulos [6] is equal to (2.6) without the premultiplication by $(\mathbf{V}_1 \otimes \mathbf{V}_2)$. The matrix $\mathbf{U}$ in (2.6) has $I(I-1)J(J-1)/4$ rows and $R(R-1)/2$ columns and condition (1.6) is equivalent to $\mathbf{U}$ being a square or vertical matrix, which is necessary for full column rank.

Next, we present two lemmas that we need in our analysis. The first lemma proves a relation between $\mathbf{A}$ and $\widetilde{\mathbf{A}}$.

Lemma 2.1. $\mathbf{A}$ *has full column rank if and only if* $\widetilde{\mathbf{A}}$ *has full column rank.*

Proof. Lemma 4 in Stegeman et al. [21] shows that $\widetilde{\mathbf{A}}$ has linearly dependent columns if this is true for $\mathbf{A}$ itself. Hence, if $\widetilde{\mathbf{A}}$ has full column rank, then also $\mathbf{A}$ has full column rank.

The converse follows from (2.3). If $\mathbf{A}$ has full column rank, then also $(\mathbf{A} \otimes \mathbf{A})$ has full column rank. Since $\mathbf{W}$ has full column rank, it follows that also $(\mathbf{A} \otimes \mathbf{A})\mathbf{W}$ has full column rank. The premultiplication by $\mathbf{V}_1$ only deletes rows which are all-zero or identical (up to sign) to another row. Therefore, (2.3) implies that $\widetilde{\mathbf{A}}$ has full column rank. □

The next lemma states that the rank of $\mathbf{U}$ does not change when $\mathbf{A}$ and $\mathbf{B}$ are premultiplied by nonsingular matrices.

Lemma 2.2. *Let* $\mathbf{S}(I \times I)$ *and* $\mathbf{T}(J \times J)$ *be nonsingular matrices and define* $\mathbf{F} = \mathbf{SA}$ *and* $\mathbf{G} = \mathbf{TB}$. *Let* $\widetilde{\mathbf{F}}$ *and* $\widetilde{\mathbf{G}}$ *be defined analogous to* $\widetilde{\mathbf{A}}$ *and* $\widetilde{\mathbf{B}}$. *Then the rank of* $\widetilde{\mathbf{F}} \odot \widetilde{\mathbf{G}}$ *is equal to the rank of* $\mathbf{U} = \widetilde{\mathbf{A}} \odot \widetilde{\mathbf{B}}$.

Proof. Consider equation (2.6). The premultiplication by $(\mathbf{V}_1 \otimes \mathbf{V}_2)$ deletes rows which are all-zero or identical (up to sign) to another row. Hence, the rank of $\mathbf{U}$ is equal to the rank of

$$((\mathbf{A} \otimes \mathbf{A})\mathbf{W}) \odot ((\mathbf{B} \otimes \mathbf{B})\mathbf{W}). \tag{2.7}$$

Premultiplying (2.7) by the nonsingular matrix $(\mathbf{S} \otimes \mathbf{S}) \otimes (\mathbf{T} \otimes \mathbf{T})$ does not change its rank. The resulting matrix is identical to $\widetilde{\mathbf{F}} \odot \widetilde{\mathbf{G}}$ without the premultiplication by $(\mathbf{V}_1 \otimes \mathbf{V}_2)$. Hence, the rank of $\widetilde{\mathbf{F}} \odot \widetilde{\mathbf{G}}$ is equal to the rank of $\mathbf{U} = \widetilde{\mathbf{A}} \odot \widetilde{\mathbf{B}}$. $\square$

3. When A and/or B has rank equal to k-rank

Here, we explore CP uniqueness when $\mathbf{C}$ has full column rank and either $\mathbf{A}$ or $\mathbf{B}$ (or both) has rank equal to k-rank. We denote the rank of $\mathbf{A}$ as $r_{\mathbf{A}}$. In our analysis, we assume that $k_{\mathbf{A}} \geqslant 2$ and $k_{\mathbf{B}} \geqslant 2$. This is necessary for CP uniqueness, see e.g. Stegeman and Sidiropoulos [22]. Our main result is that under these conditions, Kruskal's condition (1.7) implies that $\mathbf{U}$ has full column rank.

Theorem 3.1. *Let $k_{\mathbf{C}} = R$, $k_{\mathbf{A}} \geqslant 2$ and $k_{\mathbf{B}} \geqslant 2$. Suppose $r_{\mathbf{A}} = k_{\mathbf{A}}$ or $r_{\mathbf{B}} = k_{\mathbf{B}}$ or both.*
If $k_{\mathbf{A}} + k_{\mathbf{B}} \geqslant R + 2$, then $\mathbf{U}$ has full column rank and, hence, we have CP uniqueness. $\square$

However, we are able to obtain the following stronger result in which Kruskal's condition is replaced with a weaker condition.

Proposition 3.2. *Let $k_{\mathbf{C}} = R$, $k_{\mathbf{A}} \geqslant 2$ and $k_{\mathbf{B}} \geqslant 2$. Suppose $r_{\mathbf{A}} = k_{\mathbf{A}}$ or $r_{\mathbf{B}} = k_{\mathbf{B}}$ or both.*
If $r_{\mathbf{A}} + r_{\mathbf{B}} \geqslant R + 2$, then $\mathbf{U}$ has full column rank and, hence, we have CP uniqueness. $\square$

Since the k-rank of a matrix cannot be larger than its rank, it is clear that Proposition 3.2 implies Theorem 3.1. If $r_{\mathbf{A}} > k_{\mathbf{A}}$ or $r_{\mathbf{B}} > k_{\mathbf{B}}$, the condition $r_{\mathbf{A}} + r_{\mathbf{B}} \geqslant R + 2$ is more relaxed than Kruskal's condition $k_{\mathbf{A}} + k_{\mathbf{B}} \geqslant R + 2$.

We present the proof of Proposition 3.2 in Section 3.1. In Section 3.2, we prove necessary and sufficient conditions for CP uniqueness when $k_{\mathbf{C}} = R, r_{\mathbf{A}} = k_{\mathbf{A}} = R - 1$ and $k_{\mathbf{B}} \geqslant 2$. Section 3.3 contains an illustrative example.

3.1. Proof of Proposition 3.2

We consider the case $r_{\mathbf{A}} = k_{\mathbf{A}}$. The proof for $r_{\mathbf{B}} = k_{\mathbf{B}}$ is completely analogous. Suppose first $r_{\mathbf{A}} = k_{\mathbf{A}} = R$. Premultiplying $\mathbf{A}$ by a nonsingular matrix has no influence on CP uniqueness nor on the rank of $\mathbf{U}$ (see Lemma 2.2). Thus, we may transform $\mathbf{A}$ to $\begin{bmatrix} \mathbf{I}_R \\ \mathbf{O} \end{bmatrix}$ without loss of generality. We have (up to a row permutation) that $\widetilde{\mathbf{A}} = \begin{bmatrix} \mathbf{I}_{R(R-1)/2} \\ \mathbf{O} \end{bmatrix}$ and (also up to a row permutation):

$$\mathbf{U} = \begin{bmatrix} \tilde{\mathbf{b}}_{(1,2)} & & & \\ & \tilde{\mathbf{b}}_{(1,3)} & & \\ & & \ddots & \\ & & & \tilde{\mathbf{b}}_{(R-1,R)} \\ \mathbf{O} & \cdots & \cdots & \mathbf{O} \end{bmatrix}, \tag{3.1}$$

where $\tilde{\mathbf{b}}_{(g,h)}$ denotes the column of $\widetilde{\mathbf{B}}$ involving columns g and h of $\mathbf{B}$, $1 \leqslant g < h \leqslant R$. It can be seen that $\mathbf{U}$ in (3.1) has full column rank if and only if $\widetilde{\mathbf{B}}$ does not have an all-zero column. It follows from (2.4) that this is equivalent to $k_{\mathbf{B}} \geqslant 2$, which is exactly Kruskal's condition (1.7) in this case. Moreover, it can be seen that (1.5) holds if $k_{\mathbf{B}} \geqslant 2$ and, hence, we have CP uniqueness if and only if $k_{\mathbf{B}} \geqslant 2$.

Suppose next that $r_{\mathbf{A}} = k_{\mathbf{A}} = R - 1$. Then we transform $\mathbf{A}$ to $\begin{bmatrix} \mathbf{I}_{R-1} & \mathbf{p} \\ \mathbf{O} & \mathbf{0} \end{bmatrix}$ without loss of generality. Since $k_{\mathbf{A}} = R - 1$, the $(R-1)$-vector $\mathbf{p}$ does not contain any zeros. For $R = 5$ the matrix $\mathbf{U}$ has the following structure (up to a row permutation):

$$\begin{bmatrix} \tilde{\mathbf{b}}_{(1,2)} & & & p_2\tilde{\mathbf{b}}_{(1,5)} & & & -p_1\tilde{\mathbf{b}}_{(2,5)} & & & \\ & \tilde{\mathbf{b}}_{(1,3)} & & p_3\tilde{\mathbf{b}}_{(1,5)} & & & & & -p_1\tilde{\mathbf{b}}_{(3,5)} & \\ & & \tilde{\mathbf{b}}_{(1,4)} & p_4\tilde{\mathbf{b}}_{(1,5)} & & & & & & -p_1\tilde{\mathbf{b}}_{(4,5)} \\ & & & & \tilde{\mathbf{b}}_{(2,3)} & & p_3\tilde{\mathbf{b}}_{(2,5)} & & -p_2\tilde{\mathbf{b}}_{(3,5)} & \\ & & & & & \tilde{\mathbf{b}}_{(2,4)} & p_4\tilde{\mathbf{b}}_{(2,5)} & & & -p_2\tilde{\mathbf{b}}_{(4,5)} \\ & & & & & & & \tilde{\mathbf{b}}_{(3,4)} & p_4\tilde{\mathbf{b}}_{(3,5)} & -p_3\tilde{\mathbf{b}}_{(4,5)} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

The structure of $\mathbf{U}$ for general R is analogous. Let the $R(R-1)/2$-vector $\bar{\mathbf{d}}$ have elements $d_{(s,t)}$ with $1 \leqslant s < t \leqslant R$. It can be seen that for each block of $J(J-1)/2$ rows, $\mathbf{U}\bar{\mathbf{d}} = \mathbf{0}$ yields an equation of the form $d_{(s,t)}\tilde{\mathbf{b}}_{(s,t)} + d_{(s,R)}p_t\tilde{\mathbf{b}}_{(s,R)} - d_{(t,R)}p_s\tilde{\mathbf{b}}_{(t,R)} = \mathbf{0}$. From Lemma 2.1 it follows that $\tilde{\mathbf{b}}_{(s,t)}, \tilde{\mathbf{b}}_{(s,R)}$ and $\tilde{\mathbf{b}}_{(t,R)}$ are linearly independent if and only if $\mathbf{b}_s$, $\mathbf{b}_t$ and $\mathbf{b}_R$ are linearly independent. From the structure of $\mathbf{U}$ it then follows that $\mathbf{U}$ has full column rank if $k_{\mathbf{B}} \geqslant 3$, which is Kruskal's condition (1.7) in this case.

However, $k_{\mathbf{B}} \geqslant 3$ is not necessary for $\mathbf{U}$ to have full column rank. Consider again the example above with $R = 5$ and let $J \geqslant R$. Take $\mathbf{B}$ randomly sampled from a continuous distribution and set $\mathbf{b}_3 = \mathbf{b}_4 + \mathbf{b}_5$. Then $k_{\mathbf{B}} = 2$ but still $\mathbf{U}$ has full column rank. This can be seen as follows. Since $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_5$ are linearly independent, it follows from Lemma 2.1 that the first row block of $\mathbf{U}\bar{\mathbf{d}} = \mathbf{0}$ yields $d_{(1,2)} = d_{(1,5)} = d_{(2,5)} = 0$. Analogously, since the sets $\{\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5\}$, $\{\mathbf{b}_1, \mathbf{b}_4, \mathbf{b}_5\}$, $\{\mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_5\}$ and $\{\mathbf{b}_2, \mathbf{b}_4, \mathbf{b}_5\}$ are linearly independent, we obtain that all coefficients $d_{(s,t)}$ except $d_{(3,4)}$ are zero. The remaining equation is $d_{(3,4)}\tilde{\mathbf{b}}_{(3,4)} = \mathbf{0}$. Since $k_{\mathbf{B}} = 2$ the columns $\mathbf{b}_3$ and $\mathbf{b}_4$ are not proportional. By (2.4), this implies that $\tilde{\mathbf{b}}_{(3,4)} \neq \mathbf{0}$. Therefore, $d_{(3,4)} = 0$ and $\mathbf{U}$ has full column rank.

The reasoning above yields the following sufficient uniqueness condition for the case $r_{\mathbf{A}} = k_{\mathbf{A}} = R - 1$, which is more relaxed than Kruskal's condition $k_{\mathbf{B}} \geqslant 3$.

Lemma 3.3. *Let $k_{\mathbf{C}} = R$, $r_{\mathbf{A}} = k_{\mathbf{A}} = R - 1 \geqslant 2$ and $k_{\mathbf{B}} \geqslant 2$. If the set $\{\mathbf{b}_s, \mathbf{b}_t, \mathbf{b}_R\}$ is linearly independent for some s and t, then* **U** *has full column rank and, hence, we have CP uniqueness.*

Proof. Without loss of generality, we transform $\mathbf{A}$ to $\begin{bmatrix} \mathbf{I}_{R-1} & \mathbf{p} \\ \mathbf{O} & \mathbf{0} \end{bmatrix}$ and assume that $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_R\}$ is linearly independent. From $\mathbf{U}\bar{\mathbf{d}} = \mathbf{0}$ and Lemma 2.1 it follows that $d_{(1,2)} = d_{(1,R)} = d_{(2,R)} = 0$. Next, observe that for any $u \in \{3, \ldots, R-1\}$, either $\{\mathbf{b}_1, \mathbf{b}_u, \mathbf{b}_R\}$ is linearly independent, or $\{\mathbf{b}_2, \mathbf{b}_u, \mathbf{b}_R\}$ is linearly independent, or both. Then $\mathbf{U}\bar{\mathbf{d}} = \mathbf{0}$ and Lemma 2.1 imply that all coefficients $d_{(u,R)}$ are zero for $u = 3, \ldots, R-1$. The equations which are left have the form $d_{(u,v)}\tilde{\mathbf{b}}_{(u,v)} = \mathbf{0}$. Since $k_{\mathbf{B}} \geqslant 2$ it follows that also the coefficients $d_{(u,v)}$ are zero. Hence, $\mathbf{U}\bar{\mathbf{d}} = \mathbf{0}$ implies $\bar{\mathbf{d}} = \mathbf{0}$. Lemma 2.2 then implies that $\mathbf{U}$ has full column rank for any $\mathbf{A}$ and $\mathbf{B}$ satisfying the assumptions. $\square$

Next, we consider the general case $r_{\mathbf{A}} = k_{\mathbf{A}} \geqslant 2$. We transform $\mathbf{A}$ to $\begin{bmatrix} \mathbf{I}_{r_{\mathbf{A}}} & \mathbf{P} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$ without loss of generality. Since $k_{\mathbf{A}} = r_{\mathbf{A}}$, the matrix $\mathbf{P}$ does not contain any zeros. Let $\widetilde{\mathbf{P}}$ be defined in the same way as $\widetilde{\mathbf{A}}$. Then $\widetilde{\mathbf{P}}$ also does not contain any zeros. This can be seen as follows. Suppose the first element of $\tilde{\mathbf{p}}_{(1,2)}$ is zero, i.e., $p_{11}p_{22} = p_{21}p_{12}$. Then the columns $\mathbf{a}_3, \ldots, \mathbf{a}_{r_{\mathbf{A}}}, \mathbf{p}_1, \mathbf{p}_2$ are linearly dependent, where $\mathbf{a}_i$ denotes column i of $\mathbf{I}_{r_{\mathbf{A}}}$. This implies $k_{\mathbf{A}} \leq r_{\mathbf{A}} - 1$, which is a contradiction.

Let $\mathbf{B} = [\mathbf{B}_1|\mathbf{B}_2]$, where $\mathbf{B}_1$ has $r_{\mathbf{A}}$ columns and $\mathbf{B}_2$ has $R - r_{\mathbf{A}}$ columns. We have $\mathbf{U} = [\mathbf{U}_1|\widetilde{\mathbf{P}} \odot \widetilde{\mathbf{B}}_2]$, where $\mathbf{U}_1$ is the (column-wise) Khatri–Rao product of the columns (g,h) of $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{B}}$ with $1 \leqslant g \leqslant r_{\mathbf{A}}$ and $g < h \leqslant R$. For $R = 6$ and $r_{\mathbf{A}} = 4$, the structure of $\widetilde{\mathbf{A}}$ is as follows (up to a row permutation):

$$\begin{bmatrix} 1 & & & p_{21} & p_{22} & & & -p_{11} & -p_{12} & & & & & & \tilde{p}_{12} \\ & 1 & & p_{31} & p_{32} & & & & & & -p_{11} & -p_{12} & & & \tilde{p}_{13} \\ & & 1 & p_{41} & p_{42} & & & & & & & & -p_{11} & -p_{12} & \tilde{p}_{14} \\ & & & & & 1 & & p_{31} & p_{32} & & -p_{21} & -p_{22} & & & \tilde{p}_{23} \\ & & & & & & 1 & p_{41} & p_{42} & & & & -p_{21} & -p_{22} & \tilde{p}_{24} \\ & & & & & & & & & 1 & p_{41} & p_{42} & -p_{31} & -p_{32} & \tilde{p}_{34} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

where $\tilde{p}_{ij} = p_{i1}p_{j2} - p_{j1}p_{i2}$. The structure for general R and $r_{\mathbf{A}}$ is analogous. It can be seen that for each block of $J(J-1)/2$ rows, $\mathbf{U}\bar{\mathbf{d}} = \mathbf{0}$ yields an equation of the form

$$\begin{aligned} d_{(s,t)}\tilde{\mathbf{b}}_{(s,t)} &+ \sum_{h=r_{\mathbf{A}}+1}^{R} d_{(s,h)}p_{t,(h-r_{\mathbf{A}})}\tilde{\mathbf{b}}_{(s,h)} - \sum_{h=r_{\mathbf{A}}+1}^{R} d_{(t,h)}p_{s,(h-r_{\mathbf{A}})}\tilde{\mathbf{b}}_{(t,h)} \\ &+ \sum_{g=r_{\mathbf{A}}+1}^{R}\sum_{h=g+1}^{R} d_{(g,h)}(p_{sg}p_{th} - p_{tg}p_{sh})\tilde{\mathbf{b}}_{(g,h)} = \mathbf{0}, \end{aligned} \tag{3.2}$$

where $1 \leqslant s < t \leqslant r_{\mathbf{A}}$ and $(p_{sg}p_{th} - p_{tg}p_{sh})$ is an element of $\widetilde{\mathbf{P}}$. Lemma 2.1 implies that the columns of $\widetilde{\mathbf{B}}$ in (3.2) are linearly independent if and only if $\{\mathbf{b}_s, \mathbf{b}_t, \mathbf{B}_2\}$ is linearly independent. Since all elements of $\mathbf{P}$ and $\widetilde{\mathbf{P}}$ are nonzero, it follows that $\mathbf{U}\bar{\mathbf{d}} = \mathbf{0}$ implies $\bar{\mathbf{d}} = \mathbf{0}$ when $\{\mathbf{b}_s, \mathbf{b}_t, \mathbf{B}_2\}$ is linearly independent for all $1 \leqslant s < t \leqslant r_{\mathbf{A}}$. The latter is implied by Kruskal's condition (1.7) which is $k_{\mathbf{B}} \geqslant R - r_{\mathbf{A}} + 2$ in this case.

Next, we show that the condition $k_{\mathbf{B}} \geqslant R - r_{\mathbf{A}} + 2$ can be relaxed. We have the following generalization of Lemma 3.3.

Lemma 3.4. *Let $k_{\mathbf{C}} = R$, $r_{\mathbf{A}} = k_{\mathbf{A}} \geqslant 2$ and $k_{\mathbf{B}} \geqslant 2$. If the set $\{\mathbf{b}_s, \mathbf{b}_t, \mathbf{B}_2\}$ is linearly independent for some s and t, then $\mathbf{U}$ has full column rank and, hence, we have CP uniqueness.*

Proof. Without loss of generality we transform $\mathbf{A}$ to $\begin{bmatrix} \mathbf{I}_{r_{\mathbf{A}}} & \mathbf{P} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$ and assume that $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{B}_2\}$ is linearly independent. We know that $\mathbf{U}\bar{\mathbf{d}} = \mathbf{0}$ yields equations (3.2) for $1 \leqslant s < t \leqslant r_{\mathbf{A}}$. As explained above, the linear independence of $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{B}_2\}$ together with Lemma 2.1 implies that $d_{(1,2)} = 0$, $d_{(1,h)} = d_{(2,h)} = 0$ for all $h = r_{\mathbf{A}} + 1, \ldots, R$, and $d_{(g,h)} = 0$ for all $r_{\mathbf{A}} + 1 \leqslant g < h \leqslant R$.

Observe that for any $u \in \{3, \ldots, r_{\mathbf{A}}\}$, either $\{\mathbf{b}_1, \mathbf{b}_u, \mathbf{B}_2\}$ is linearly independent, or $\{\mathbf{b}_2, \mathbf{b}_u, \mathbf{B}_2\}$ is linearly independent, or both. It then follows from (3.2) and Lemma 2.1 that all coefficients $d_{(u,h)}$ are zero for $u = 3, \ldots, r_{\mathbf{A}}$ and $h = r_{\mathbf{A}} + 1, \ldots, R$. The equations that are left have the form $d_{(u,v)}\tilde{\mathbf{b}}_{(u,v)} = \mathbf{0}$. Since $k_{\mathbf{B}} \geqslant 2$ it follows that also the coefficients $d_{(u,v)}$ are zero. Lemma 2.2 then implies that $\mathbf{U}$ has full column rank for any $\mathbf{A}$ and $\mathbf{B}$ satisfying the assumptions. □

Note that, in the proof of Lemma 3.4, the transformation of $\mathbf{A}$ to $\begin{bmatrix} \mathbf{I}_{r_{\mathbf{A}}} & \mathbf{P} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$ works for any order of the columns of $\mathbf{A}$. This implies that the partition of $\mathbf{B}$ into $\mathbf{B}_1$ and $\mathbf{B}_2$ is immaterial. In Lemma 3.4, we just need one group of $R - r_{\mathbf{A}} + 2$ linearly independent columns in $\mathbf{B}$. This yields the following result.

Lemma 3.5. *Let $k_{\mathbf{C}} = R$, $r_{\mathbf{A}} = k_{\mathbf{A}} \geqslant 2$ and $k_{\mathbf{B}} \geqslant 2$.*

If $r_{\mathbf{A}} + r_{\mathbf{B}} \geqslant R + 2$, then $\mathbf{U}$ has full column rank and, hence, we have CP uniqueness. □

The proof of Proposition 3.2 is complete by observing that $\mathbf{A}$ and $\mathbf{B}$ are interchangeable in Lemma 3.5.

3.2. Necessary and sufficient CP uniqueness conditions

Above, we saw that if $r_{\mathbf{A}} = k_{\mathbf{A}} = R \geqslant 2$, then $k_{\mathbf{B}} \geqslant 2$ is necessary and sufficient for CP uniqueness. In this section, we present a necessary and sufficient CP uniqueness condition for the case $r_{\mathbf{A}} = k_{\mathbf{A}} = R - 1 \geqslant 2$. The case $r_{\mathbf{B}} = k_{\mathbf{B}} = R - 1 \geqslant 2$ is analogous. As before, we assume that **C** has full column rank.

First, we show that the condition of Lemma 3.5 is also necessary for **U** to have full column rank if $r_{\mathbf{A}} = k_{\mathbf{A}} = R - 1 \geqslant 2$. Numerical experiments show that the condition is not necessary if $r_{\mathbf{A}} = k_{\mathbf{A}} < R - 1$.

Lemma 3.6. *Let $k_{\mathbf{C}} = R$, $r_{\mathbf{A}} = k_{\mathbf{A}} = R - 1 \geqslant 2$ and $k_{\mathbf{B}} \geqslant 2$.*
*Then **U** has full column rank if and only if $r_{\mathbf{B}} \geqslant 3$.*

Proof. Sufficiency follows from Lemma 3.5. We transform **A** to $\begin{bmatrix} \mathbf{I}_{R-1} & \mathbf{p} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ without loss of generality (see Lemma 2.2). The $(R-1)$-vector $\mathbf{p}$ does not contain any zeros. Let $r_{\mathbf{B}} = 2$. Then every group of three columns of **B** is linearly dependent. Since $k_{\mathbf{B}} \geqslant 2$ every two columns of **B** are linearly independent. Hence, there holds in particular

$$\mathbf{b}_R = \alpha_1^{(s)} \mathbf{b}_1 + \alpha_2^{(s)} \mathbf{b}_s \quad \text{for } s = 2, \ldots, R-1, \tag{3.3}$$

where $\alpha_1^{(s)} \neq 0$ and $\alpha_2^{(s)} \neq 0$. This implies that

$$\begin{vmatrix} b_{i1} & b_{iR} \\ b_{j1} & b_{jR} \end{vmatrix} = \alpha_2^{(s)} \begin{vmatrix} b_{i1} & b_{is} \\ b_{j1} & b_{js} \end{vmatrix} \quad \text{for } 1 \leqslant i < j \leqslant J, \tag{3.4}$$

which is equivalent to $\tilde{\mathbf{b}}_{(1,R)} = \alpha_2^{(s)} \tilde{\mathbf{b}}_{(1,s)}$ for $s = 2, \ldots, R-1$. It follows that

$$p_2 \alpha_2^{(2)} \begin{pmatrix} \tilde{\mathbf{b}}_{(1,2)} \\ \mathbf{0} \\ \vdots \\ \vdots \\ \mathbf{0} \end{pmatrix} + p_3 \alpha_2^{(3)} \begin{pmatrix} \mathbf{0} \\ \tilde{\mathbf{b}}_{(1,3)} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix} + \cdots + p_{R-1} \alpha_2^{(R-1)} \begin{pmatrix} \mathbf{0} \\ \vdots \\ \vdots \\ \mathbf{0} \\ \tilde{\mathbf{b}}_{(1,R-1)} \end{pmatrix} = \begin{pmatrix} p_2 \tilde{\mathbf{b}}_{(1,R)} \\ p_3 \tilde{\mathbf{b}}_{(1,R)} \\ \vdots \\ \vdots \\ p_{R-1} \tilde{\mathbf{b}}_{(1,R)} \end{pmatrix}. \tag{3.5}$$

Eq. (3.5) shows the nonzero parts of the first $R-1$ columns of **U** and, hence, their linear dependence. This completes the proof. $\square$

Lemma 3.6 shows that if $r_{\mathbf{A}} = k_{\mathbf{A}} = R - 1 \geqslant 2$, then a necessary condition for non-uniqueness is $r_{\mathbf{B}} \leqslant 2$. Next, we show that $r_{\mathbf{B}} = 2$ is sufficient for non-uniqueness.

Lemma 3.7. *Let $k_{\mathbf{C}} = R$, $r_{\mathbf{A}} = k_{\mathbf{A}} = R - 1 \geqslant 2$ and $k_{\mathbf{B}} \geqslant 2$.*
Then we have CP uniqueness if and only if $r_{\mathbf{B}} \geqslant 3$.

Proof. Without loss of generality, we assume that **A** is transformed to $\begin{bmatrix} \mathbf{I}_{R-1} & \mathbf{p} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$. From Lemma 3.6 it follows that non-uniqueness is only possible if $r_{\mathbf{B}} = 2$. Next, we show that this is also sufficient for non-uniqueness.

Let $R = 3$ and $r_{\mathbf{B}} = 2$. Since $k_{\mathbf{B}} \geqslant 2$, we have $\mathbf{b}_3 = \alpha \mathbf{b}_1 + \beta \mathbf{b}_2$ with $\alpha \neq 0$ and $\beta \neq 0$. This implies $\tilde{\mathbf{b}}_{(1,3)} = \beta \tilde{\mathbf{b}}_{(1,2)}$ and $\tilde{\mathbf{b}}_{(2,3)} = -\alpha \tilde{\mathbf{b}}_{(1,2)}$. Eq. (1.5) reads as

$$d_1 d_2 \tilde{\mathbf{b}}_{(1,2)} + d_1 d_3 p_2 \tilde{\mathbf{b}}_{(1,3)} - d_2 d_3 p_1 \tilde{\mathbf{b}}_{(2,3)} = \mathbf{0}, \tag{3.6}$$

which is equivalent to

$$[d_1d_2 + d_1d_3p_2\beta + d_2d_3p_1\alpha]\tilde{\mathbf{b}}_{(1,2)} = \mathbf{0}. \tag{3.7}$$

Since $k_{\mathbf{B}} \geqslant 2$, the coefficient of $\tilde{\mathbf{b}}_{(1,2)}$ in (3.7) must be zero. This is always possible with nonzero d_1, d_2 and d_3. Hence, if $r_{\mathbf{B}} = 2$, condition (1.5) does not hold and we have non-uniqueness.

Next, let $R = 4$ and $r_{\mathbf{B}} = 2$. Since $k_{\mathbf{B}} \geqslant 2$ any two columns of $\mathbf{B}$ span its column space. We write $\mathbf{b}_4 = \alpha\mathbf{b}_1 + \beta\mathbf{b}_2$ and $\mathbf{b}_3 = \gamma\mathbf{b}_1 + \delta\mathbf{b}_2$, with $\alpha\beta\gamma\delta \neq 0$. This implies $\mathbf{b}_4 = (\alpha - \beta\gamma/\delta)\mathbf{b}_1 + (\beta/\delta)\mathbf{b}_3$ and $\mathbf{b}_4 = (\beta - \alpha\delta/\gamma)\mathbf{b}_2 + (\alpha/\gamma)\mathbf{b}_3$. For the columns of $\tilde{\mathbf{B}}$, we obtain the following relations.

$$\tilde{\mathbf{b}}_{(1,4)} = \beta\tilde{\mathbf{b}}_{(1,2)} \quad \tilde{\mathbf{b}}_{(2,4)} = -\alpha\tilde{\mathbf{b}}_{(1,2)}, \tag{3.8}$$

$$\tilde{\mathbf{b}}_{(1,4)} = (\beta/\delta)\tilde{\mathbf{b}}_{(1,3)} \quad \tilde{\mathbf{b}}_{(3,4)} = -(\alpha - \beta\gamma/\delta)\tilde{\mathbf{b}}_{(1,3)}, \tag{3.9}$$

$$\tilde{\mathbf{b}}_{(2,4)} = (\alpha/\gamma)\tilde{\mathbf{b}}_{(2,3)} \quad \tilde{\mathbf{b}}_{(3,4)} = -(\beta - \alpha\delta/\gamma)\tilde{\mathbf{b}}_{(2,3)}. \tag{3.10}$$

Eq. (1.5) yields the equations $d_sd_t\tilde{\mathbf{b}}_{(s,t)} + d_sd_4p_t\tilde{\mathbf{b}}_{(s,4)} - d_td_4p_s\tilde{\mathbf{b}}_{(t,4)} = \mathbf{0}$ for $1 \leqslant s < t \leqslant 3$. Substituting (3.8)–(3.10) and using $k_{\mathbf{B}} \geqslant 2$ as above, we obtain

$$d_1d_2 + d_1d_4p_2\beta + d_2d_4p_1\alpha = 0, \tag{3.11}$$

$$d_1d_3 + d_1d_4p_3(\beta/\delta) + d_3d_4p_1(\alpha - \beta\gamma/\delta) = 0, \tag{3.12}$$

$$d_2d_3 + d_2d_4p_3(\alpha/\gamma) + d_3d_4p_2(\beta - \alpha\delta/\gamma) = 0. \tag{3.13}$$

If $d_2 + d_4p_2\beta \neq 0$ and $d_3 + d_4p_3(\beta/\delta) \neq 0$, it follows from (3.11)–(3.12) that

$$d_2[d_3d_4p_1(\beta\gamma/\delta) + d_4^2p_1p_3(\alpha\beta/\delta)] = d_3d_4^2p_1p_2\beta(\alpha - \beta\gamma/\delta). \tag{3.14}$$

Eq. (3.14) is identical to (3.13) multiplied by $d_4p_1(\beta\gamma/\delta)$. Hence, Eqs. (3.11) and (3.12) imply Eq. (3.13) if d_4 is nonzero (and $d_2 + d_4p_2\beta \neq 0$ and $d_3 + d_4p_3(\beta/\delta) \neq 0$). The system (1.5) can be solved by choosing appropriate nonzero d_1 and d_4. Then the value of d_2 follows from (3.11) and d_3 follows from (3.12). This implies that if $r_{\mathbf{B}} = 2$, condition (1.5) does not hold and we have non-uniqueness.

The proof for the general case $R \geqslant 4$ and $r_{\mathbf{B}} = 2$ is similar. We refer to the equation of (1.5) starting with d_sd_t as equation (s,t). From the proof for $R = 4$ it follows that equations (s,t) and (s,u) imply equation (t,u). This implies that we only need to consider the equations $(1,t)$ for $2 \leqslant t \leqslant R-1$. The system (1.5) can be solved by choosing appropriate nonzero d_1 and d_R. Then the value of d_t follows from equation $(1,t)$, $2 \leqslant t \leqslant R-1$. Hence, we have non-uniqueness. □

3.3. An example

Lemma 3.7 shows that the condition in Proposition 3.2 and $\mathbf{U}$ having full column rank are both also necessary for CP uniqueness when $r_{\mathbf{A}} = k_{\mathbf{A}} = R - 1 \geqslant 2$ and $k_{\mathbf{B}} \geqslant 2$.

In general, however, the condition of Proposition 3.2 and $\mathbf{U}$ having full column rank are not necessary for CP uniqueness. A counterexample is provided by Stegeman and Ten Berge [20]. In this example,

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 & 5 \end{bmatrix}, \tag{3.15}$$

and $\mathbf{C} = \mathbf{I}_5$. We have $r_{\mathbf{A}} = k_{\mathbf{A}} = 3$ and $r_{\mathbf{B}} = k_{\mathbf{B}} = 3$. The condition in Proposition 3.2 does not hold, since $r_{\mathbf{A}} + r_{\mathbf{B}} = 6$ is less than $R + 2 = 7$. Also, the matrix $\mathbf{U}$ is 9×10 and cannot have full column rank. Yet, the CP solution is shown to be unique by Stegeman and Ten Berge [20]. This can also be done using the necessary and sufficient condition (1.5) of Jiang and Sidiropoulos [6].

4. When A and B have rank larger than k-rank

Here, we explore CP uniqueness when $\mathbf{C}$ has full column rank and both $\mathbf{A}$ and $\mathbf{B}$ have rank larger than k-rank. Our main result is that under these conditions, Kruskal's condition (1.7) implies that $\mathbf{U}$ has full column rank.

Theorem 4.1. *Let $k_{\mathbf{C}} = R$, $r_{\mathbf{A}} > k_{\mathbf{A}} \geqslant 2$ and $r_{\mathbf{B}} > k_{\mathbf{B}} \geqslant 2$.*
If $k_{\mathbf{A}} + k_{\mathbf{B}} \geqslant R + 2$, then $\mathbf{U}$ has full column rank and, hence, we have CP uniqueness. □

Theorem 3.1 and Theorem 4.1 together show that if $\mathbf{C}$ has full column rank and $\mathbf{A}$ and $\mathbf{B}$ have k-rank at least 2, then Kruskal's condition $k_{\mathbf{A}} + k_{\mathbf{B}} \geqslant R + 2$ implies that $\mathbf{U}$ has full column rank. The latter is the sufficient uniqueness condition of Jiang and Sidiropoulos [6] and De Lathauwer [2].

Instead of proving Theorem 4.1, we will show the following stronger result. This is the analogue of Proposition 3.2.

Proposition 4.2. *Let $k_{\mathbf{C}} = R$, $r_{\mathbf{A}} > k_{\mathbf{A}} \geqslant 2$ and $r_{\mathbf{B}} > k_{\mathbf{B}} \geqslant 2$.*
If $\max(r_{\mathbf{A}} + k_{\mathbf{B}}, k_{\mathbf{A}} + r_{\mathbf{B}}) \geqslant R + 2$, then $\mathbf{U}$ has full column rank and, hence, we have CP uniqueness. □

The condition of Proposition 4.2 is more relaxed than Kruskal's condition $k_{\mathbf{A}} + k_{\mathbf{B}} \geqslant R + 2$. Hence, Proposition 4.2 implies Theorem 4.1.

We present the proof of Proposition 4.2 in Section 4.1. In Section 4.2, we prove necessary and sufficient conditions for CP uniqueness when $k_{\mathbf{C}} = R$, $R - 1 = r_{\mathbf{A}} > k_{\mathbf{A}} \geqslant 2$ and $k_{\mathbf{B}} \geqslant 2$. Section 4.3 contains an illustrative example.

4.1. Proof of Proposition 4.2

We only prove the uniqueness condition $r_{\mathbf{A}} + k_{\mathbf{B}} \geqslant R + 2$. The proof of $k_{\mathbf{A}} + r_{\mathbf{B}} \geqslant R + 2$ is completely analogous. We assume that $R - 1 \geqslant r_{\mathbf{A}} > k_{\mathbf{A}} \geqslant 2$ and $k_{\mathbf{B}} \geqslant 2$. Note that $r_{\mathbf{A}} = R$ implies $k_{\mathbf{A}} = R$, and this case has been discussed in Section 3. We simultaneously reorder the columns of $\mathbf{A}$ and $\mathbf{B}$ such that the first $r_{\mathbf{A}}$ columns of $\mathbf{A}$ are linearly independent. Then we transform $\mathbf{A}$ to $\begin{bmatrix} \mathbf{I}_{r_{\mathbf{A}}} & \mathbf{P} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$ without loss of generality. Since $k_{\mathbf{A}} < r_{\mathbf{A}}$ the matrix $\mathbf{P}$ may contain zero elements. Also, the matrix $\widetilde{\mathbf{P}}$ may contain zero elements. As in Section 3, we partition $\mathbf{B} = [\mathbf{B}_1 | \mathbf{B}_2]$, where $\mathbf{B}_1$ has $r_{\mathbf{A}}$ columns and $\mathbf{B}_2$ has $R - r_{\mathbf{A}}$ columns, and we write $\mathbf{U} = [\mathbf{U}_1 | \widetilde{\mathbf{P}} \odot \widetilde{\mathbf{B}}_2]$. The structure of $\mathbf{U}$ is the same as in Section 3, except that some elements of $\mathbf{P}$ and $\widetilde{\mathbf{P}}$ may be zero. Using this fact, we obtain the following analogue of Lemma 3.4.

Lemma 4.3. *Let $k_{\mathbf{C}} = R$, $R - 1 \geqslant r_{\mathbf{A}} > k_{\mathbf{A}} \geqslant 2$ and $k_{\mathbf{B}} \geqslant 2$. If the set $\{\mathbf{b}_s, \mathbf{b}_t, \mathbf{B}_2\}$ is linearly independent for some s and t, and rows s and t of $\mathbf{P}$ and row (s,t) of $\widetilde{\mathbf{P}}$ contain no zeros, then $\mathbf{U}$ has full column rank and, hence, we have CP uniqueness.*

Proof. Without loss of generality we transform $\mathbf{A}$ to $\begin{bmatrix} \mathbf{I}_{r_{\mathbf{A}}} & \mathbf{P} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$ and assume that $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{B}_2\}$ is linearly independent, and rows 1 and 2 of $\mathbf{P}$ and row $(1,2)$ of $\widetilde{\mathbf{P}}$ contain no zeros. The proof is analogous to the proof of Lemma 3.4. We know that $\mathbf{U}\bar{\mathbf{d}} = \mathbf{0}$ yields equations (3.2) for $1 \leqslant s < t \leqslant r_{\mathbf{A}}$. In the equation for $(s,t) = (1,2)$ the elements of $\mathbf{P}$ and $\widetilde{\mathbf{P}}$ are nonzero by assumption. As in the proof of Lemma 3.4, the linear independence of $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{B}_2\}$ together with Lemma 2.1 implies that $d_{(1,2)} = 0$, $d_{(1,h)} = d_{(2,h)} = 0$ for all $h = r_{\mathbf{A}} + 1, \ldots, R$, and $d_{(g,h)} = 0$ for all $r_{\mathbf{A}} + 1 \leqslant g < h \leqslant R$.

Observe that for any $u \in \{3, \ldots, r_{\mathbf{A}}\}$, either $\{\mathbf{b}_1, \mathbf{b}_u, \mathbf{B}_2\}$ is linearly independent, or $\{\mathbf{b}_2, \mathbf{b}_u, \mathbf{B}_2\}$ is linearly independent, or both. After deleting the terms with a coefficient of $\bar{\mathbf{d}}$ that is zero, the equations (3.2) for $(s,t) = (s,u)$ with $s = 1, 2$ and $u = 3, \ldots, r_{\mathbf{A}}$ are

$$d_{(s,u)} \tilde{\mathbf{b}}_{(s,u)} - \sum_{h=r_{\mathbf{A}}+1}^{R} d_{(u,h)} p_{s,(h-r_{\mathbf{A}})} \tilde{\mathbf{b}}_{(u,h)} = \mathbf{0}, \tag{4.1}$$

where the elements $p_{s,(h-r_{\mathbf{A}})}$ are nonzero by assumption. When either $\{\mathbf{b}_1, \mathbf{b}_u, \mathbf{B}_2\}$ is linearly independent, or $\{\mathbf{b}_2, \mathbf{b}_u, \mathbf{B}_2\}$, or both, it follows from (4.1) and Lemma 2.1 that all coefficients $d_{(u,h)}$ are zero for $u = 3, \ldots, r_{\mathbf{A}}$ and $h = r_{\mathbf{A}} + 1, \ldots, R$. The equations of $\mathbf{U}\bar{\mathbf{d}} = \mathbf{0}$ that are left have the form $d_{(u,v)} \tilde{\mathbf{b}}_{(u,v)} = \mathbf{0}$. Since $k_{\mathbf{B}} \geqslant 2$ it follows that also the coefficients $d_{(u,v)}$ are zero. Lemma 2.2 then implies that $\mathbf{U}$ has full column rank for any $\mathbf{A}$ and $\mathbf{B}$ satisfying the assumptions. □

As in the case where $r_{\mathbf{A}} = k_{\mathbf{A}}$, each row block of $\mathbf{U}\bar{\mathbf{d}} = \mathbf{0}$ yields an Eq. (3.2) featuring a linear combination of the columns $\tilde{\mathbf{b}}_{(s,t)}$ for some $1 \leqslant s < t \leqslant r_{\mathbf{A}}$, columns $\tilde{\mathbf{b}}_{(s,h)}$ and $\tilde{\mathbf{b}}_{(t,h)}$ for all $h = r_{\mathbf{A}} + 1, \ldots, R$, and columns $\tilde{\mathbf{b}}_{(g,h)}$ for all $r_{\mathbf{A}} + 1 \leqslant g < h \leqslant R$. The difference is that now some of these columns may not appear in the linear combinations due to zero elements of $\mathbf{P}$ or $\widetilde{\mathbf{P}}$. However, it remains true that these columns are linearly independent if and only if $\{\mathbf{b}_s, \mathbf{b}_t, \mathbf{B}_2\}$ is linearly independent (see Lemma 2.1). The latter is implied by $k_{\mathbf{B}} \geqslant R - r_{\mathbf{A}} + 2$. This yields the following analogue of Lemma 3.5.

Lemma 4.4. *Let $k_{\mathbf{C}} = R$, $r_{\mathbf{A}} > k_{\mathbf{A}} \geqslant 2$ and $k_{\mathbf{B}} \geqslant 2$.*
If $r_{\mathbf{A}} + k_{\mathbf{B}} \geqslant R + 2$, then $\mathbf{U}$ has full column rank and, hence, we have CP uniqueness. □

The proof of Proposition 4.2 is complete by interchanging **A** and **B** in the proof above.

The uniqueness condition of Lemma 4.3 is more relaxed than the one in Lemma 4.4. This is because, contrary to the case $r_{\mathbf{A}} = k_{\mathbf{A}}$, the transformation of **A** to $\begin{bmatrix} \mathbf{I}_{r_{\mathbf{A}}} & \mathbf{P} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$ depends on the order of the columns of **A**. The weaker uniqueness condition of Lemma 4.3 allows more subtle interactions between **A** and **B**. In the even weaker condition of **U** having full column rank, the matrices **A** and **B** are even more intertwined.

An example where the condition of Lemma 4.3 holds but not the conditions in Lemma 4.4 and Proposition 4.2 is (4.18) in Section 4.3 with $a_1 = b_2 = 0$. Then $r_{\mathbf{A}} + k_{\mathbf{B}} = k_{\mathbf{A}} + r_{\mathbf{B}} = 5$ and $R + 2 = 6$. But the condition in Lemma 4.3 holds as is shown in Section 4.3.

4.2. Necessary and sufficient CP uniqueness conditions

Here, we prove that the condition in Lemma 4.3 is also necessary for CP uniqueness if $R - 1 = r_{\mathbf{A}} > k_{\mathbf{A}} \geqslant 2$. First, however, we show that if $R - 1 = r_{\mathbf{A}} > k_{\mathbf{A}} \geqslant 2$, then the condition in Lemma 4.3 is necessary for **U** to have full column rank. This is the analogue of Lemma 3.6. Numerical experiments show that the condition is not necessary for **U** to have full column rank if $R - 1 > r_{\mathbf{A}} > k_{\mathbf{A}} \geqslant 2$. Note that for $r_{\mathbf{A}} = R - 1$ the matrix **P** is an $(R-1)$-vector **p** and $\widetilde{\mathbf{P}}$ does not exist. We assume that the columns of **A** and **B** are simultaneously permuted such that the first $R - 1$ columns of **A** are linearly independent. As before, the analysis for $R - 1 = r_{\mathbf{B}} > k_{\mathbf{B}} \geqslant 2$ is completely analogous.

Lemma 4.5. *Let $k_{\mathbf{C}} = R$, $R - 1 = r_{\mathbf{A}} > k_{\mathbf{A}} \geqslant 2$ and $k_{\mathbf{B}} \geqslant 2$.*
*Then **U** has full column rank if and only if for some s and t, the set $\{\mathbf{b}_s, \mathbf{b}_t, \mathbf{b}_R\}$ is linearly independent and elements p_s and p_t of **p** are nonzero.*

Proof. Sufficiency follows from Lemma 4.3. We transform **A** to $\begin{bmatrix} \mathbf{I}_{R-1} & \mathbf{p} \\ \mathbf{O} & \mathbf{0} \end{bmatrix}$ without loss of generality (see Lemma 2.2). For $1 \leqslant s < t \leqslant R - 1$, we define the following sets:

$$\mathcal{D} = \{(s,t) : \{\mathbf{b}_s, \mathbf{b}_t, \mathbf{b}_R\} \text{ linearly dependent}\}, \tag{4.2}$$

$$\mathcal{I} = \{(s,t) : \{\mathbf{b}_s, \mathbf{b}_t, \mathbf{b}_R\} \text{ linearly independent}\}. \tag{4.3}$$

The equations of $\mathbf{U}\bar{\mathbf{d}} = \mathbf{0}$ are given by

$$d_{(s,t)}\tilde{\mathbf{b}}_{(s,t)} + d_{(s,R)}p_t\tilde{\mathbf{b}}_{(s,R)} - d_{(t,R)}p_s\tilde{\mathbf{b}}_{(t,R)} = \mathbf{0} \tag{4.4}$$

for all $1 \leqslant s < t \leqslant R - 1$. In the proof below, we assume that the condition of the lemma does not hold and show that **U** has linearly dependent columns.

First, we consider the case $\mathcal{I} = \emptyset$, i.e. all sets $\{\mathbf{b}_s, \mathbf{b}_t, \mathbf{b}_R\}$ are linearly dependent. As in the proof of Lemma 3.6 we have in particular

$$\mathbf{b}_R = \alpha_1^{(t)}\mathbf{b}_1 + \alpha_2^{(t)}\mathbf{b}_t \quad \text{for } t = 2, \ldots, R-1, \tag{4.5}$$

where $\alpha_1^{(t)} \neq 0$ and $\alpha_2^{(t)} \neq 0$. The equations of $\mathbf{U}\bar{\mathbf{d}} = \mathbf{0}$ with coefficients $d_{(1,t)}$ are

$$d_{(1,t)}\tilde{\mathbf{b}}_{(1,t)} + d_{(1,R)}p_t\tilde{\mathbf{b}}_{(1,R)} - d_{(t,R)}p_1\tilde{\mathbf{b}}_{(t,R)} = \mathbf{0} \quad \text{for } t = 2, \ldots, R-1. \tag{4.6}$$

Eq. (4.5) implies $\tilde{\mathbf{b}}_{(1,R)} = \alpha_2^{(t)}\tilde{\mathbf{b}}_{(1,t)}$ and $\tilde{\mathbf{b}}_{(t,R)} = -\alpha_1^{(t)}\tilde{\mathbf{b}}_{(1,t)}$. Hence, (4.6) becomes

$$[d_{(1,t)} + d_{(1,R)}p_t\alpha_2^{(t)} + d_{(t,R)}p_1\alpha_1^{(t)}]\tilde{\mathbf{b}}_{(1,t)} = \mathbf{0} \quad \text{for } t = 2,\dots,R-1. \tag{4.7}$$

Since $k_{\mathbf{B}} \geqslant 2$, the coefficient of $\tilde{\mathbf{b}}_{(1,t)}$ in (4.7) must be zero. Since $k_{\mathbf{A}} \geqslant 2$ we have that $p_t \neq 0$ for some $t \in \{2,\dots,R-1\}$. Otherwise $\mathbf{p}$ is proportional to a column of $\mathbf{I}_{R-1}$ and, hence, $k_{\mathbf{A}} = 1$ would hold. It follows from (4.7) that we may take $d_{(1,R)}$ nonzero. Each equation is then solved by an appropriate choice of $d_{(1,t)}$, where the values of $d_{(t,R)}$ are determined by the other equations in $\mathbf{U}\bar{\mathbf{d}} = \mathbf{0}$. Therefore, $\mathbf{U}\bar{\mathbf{d}} = \mathbf{0}$ does not imply $\bar{\mathbf{d}} = \mathbf{0}$.

In the remaining part of the proof we assume $\mathcal{I} \neq \emptyset$. It follows from (4.4) and Lemma 2.1 that for $(s,t) \in \mathcal{I}$

$$d_{(s,t)} = d_{(s,R)}p_t = d_{(t,R)}p_s = p_sp_t = 0, \tag{4.8}$$

where $p_sp_t = 0$ holds because we assume that the condition of the lemma does not hold.

Suppose $\mathcal{D} = \emptyset$. Then (4.8), and in particular $p_sp_t = 0$, holds for all $1 \leqslant s < t \leqslant R-1$. It follows that $\mathbf{p}$ contains at most one nonzero element, which implies $k_{\mathbf{A}} \leqslant 1$. This contradicts $k_{\mathbf{A}} \geqslant 2$. In the remaining part of the proof we assume $\mathcal{D} \neq \emptyset$.

Let $(s,t) \in \mathcal{I}$. Without loss of generality we set $s = 1$ and $t = 2$. We have (4.8) with $(s,t) = (1,2)$, i.e.

$$d_{(1,2)} = d_{(1,R)}p_2 = d_{(2,R)}p_1 = p_1p_2 = 0. \tag{4.9}$$

The other equations in $\mathbf{U}\bar{\mathbf{d}} = \mathbf{0}$ involving $\mathbf{b}_1$ and $\mathbf{b}_2$ are

$$d_{(1,u)}\tilde{\mathbf{b}}_{(1,u)} + d_{(1,R)}p_u\tilde{\mathbf{b}}_{(1,R)} - d_{(u,R)}p_1\tilde{\mathbf{b}}_{(u,R)} = \mathbf{0} \quad \text{for } u = 3,\dots,R-1, \tag{4.10}$$

$$d_{(2,u)}\tilde{\mathbf{b}}_{(2,u)} + d_{(2,R)}p_u\tilde{\mathbf{b}}_{(2,R)} - d_{(u,R)}p_2\tilde{\mathbf{b}}_{(u,R)} = \mathbf{0} \quad \text{for } u = 3,\dots,R-1. \tag{4.11}$$

Observe that for any $u \in \{3,\dots,R-1\}$, either $(1,u) \in \mathcal{I}$, or $(2,u) \in \mathcal{I}$, or both. Hence, it follows from Lemma 2.1 and (4.10)–(4.11) that $d_{(1,R)}d_{(2,R)}p_u = 0$ for $u = 3,\dots,R-1$. Note that it follows from $p_1p_2 = 0$ that $p_u = 0$ for $u = 3,\dots,R-1$ implies that $\mathbf{p}$ is proportional to a column of $\mathbf{I}_{R-1}$ and, hence, that $k_{\mathbf{A}} = 1$. Since the latter contradicts $k_{\mathbf{A}} \geqslant 2$, we have that $p_u \neq 0$ for some $u \in \{3,\dots,R-1\}$. Hence, we obtain

$$d_{(1,R)}d_{(2,R)} = 0. \tag{4.12}$$

Next, we define

$$\mathcal{S}_1 = \{u \geqslant 3 : (1,u) \in \mathcal{I}\}. \tag{4.13}$$

It follows from (4.8) that

$$d_{(1,u)} = d_{(1,R)}p_u = d_{(u,R)}p_1 = p_1p_u = 0 \quad \text{for all } u \in \mathcal{S}_1. \tag{4.14}$$

Suppose $p_1 \neq 0$ (the case $p_2 \neq 0$ is completely analogous). Then $p_2 = 0$, (4.9) implies $d_{(2,R)} = 0$, and (4.14) implies $p_u = 0$ for all $u \in \mathcal{S}_1$. Since $k_{\mathbf{B}} \geqslant 2$, it follows from (4.11) that $d_{(2,u)} = 0$ for $u = 3,\dots,R-1$. Equation (4.14) gives $d_{(1,u)} = d_{(u,R)} = 0$ for all $u \in \mathcal{S}_1$. We are left with (4.10) for $u \notin \mathcal{S}_1$. For any such u, we have $(1,u) \in \mathcal{D}$. As in (4.5) there holds

$$\mathbf{b}_R = \alpha_1^{(u)}\mathbf{b}_1 + \alpha_2^{(u)}\mathbf{b}_u \quad \text{for } u \notin \mathcal{S}_1, \tag{4.15}$$

where $\alpha_1^{(u)} \neq 0$ and $\alpha_2^{(u)} \neq 0$. This implies $\tilde{\mathbf{b}}_{(1,R)} = \alpha_2^{(u)}\tilde{\mathbf{b}}_{(1,u)}$ and $\tilde{\mathbf{b}}_{(u,R)} = -\alpha_1^{(u)}\tilde{\mathbf{b}}_{(1,u)}$. Eq. (4.10) now becomes

$$[d_{(1,u)} + d_{(1,R)}p_u\alpha_2^{(u)} + d_{(u,R)}p_1\alpha_1^{(u)}]\tilde{\mathbf{b}}_{(1,u)} = \mathbf{0} \quad \text{for } u \notin \mathcal{S}_1. \tag{4.16}$$

Since $k_{\mathbf{B}} \geqslant 2$, the coefficient of $\tilde{\mathbf{b}}_{(1,u)}$ in (4.16) must be zero. Since $k_{\mathbf{A}} \geqslant 2$, it follows from $p_u = 0$ for all $u \in \mathcal{S}_1$ that $p_u \neq 0$ for some $u \notin \mathcal{S}_1$. This implies that for any nonzero $d_{(1,R)}$, Eq. (4.16) can be solved by an appropriate choice of $d_{(1,u)}$, where the values of $d_{(u,R)}$ are determined by the other equations in $\mathbf{U}\bar{\mathbf{d}} = \mathbf{0}$. Therefore, $\mathbf{U}\bar{\mathbf{d}} = \mathbf{0}$ does not imply $\bar{\mathbf{d}} = \mathbf{0}$.

Finally, we consider the case $(1,2) \in \mathcal{I}$ with $p_1 = p_2 = 0$. As above, Eqs. (4.9)–(4.12) hold. We have $d_{(1,2)} = 0$ and without loss of generality we set $d_{(2,R)} = 0$ so that (4.12) is satisfied. Since $k_\mathbf{B} \geqslant 2$ it follows from (4.11) that $d_{(2,u)} = 0$ for $u = 3, \ldots, R-1$. As above, Eq. (4.14) should hold. In particular, $d_{(1,u)} = d_{(1,R)} p_u = 0$ for all $u \in \mathcal{S}_1$. As in (4.16) we obtain that

$$d_{(1,u)} + d_{(1,R)} p_u \alpha_2^{(u)} = 0 \quad \text{for } u \notin \mathcal{S}_1. \tag{4.17}$$

If $p_u = 0$ for all $u \in \mathcal{S}_1$, then $p_u \neq 0$ for some $u \notin \mathcal{S}_1$ (since $k_\mathbf{A} \geqslant 2$). For any nonzero $d_{(1,R)}$, Eq. (4.17) is solved by an appropriate choice of $d_{(1,u)}$. Hence, $\mathbf{U}\bar{\mathbf{d}} = \mathbf{0}$ does not imply $\bar{\mathbf{d}} = \mathbf{0}$.

If $p_u \neq 0$ for some $u \in \mathcal{S}_1$, then $d_{(1,R)} = 0$ follows from (4.14). Eq. (4.17) implies $d_{(1,u)} = 0$ for all $u \notin \mathcal{S}_1$. Now we have $d_{(1,u)} = d_{(2,u)} = 0$ for $u = 3, \ldots, R$ and $d_{(1,2)} = 0$ and $p_1 = p_2 = 0$. Hence, the Eq. (4.4) of $\mathbf{U}\bar{\mathbf{d}} = \mathbf{0}$ with $s = 1, 2$ and $s < t \leqslant R-1$ imply that the corresponding coefficients of $\bar{\mathbf{d}}$ are zero (except $d_{(t,R)}$ since $p_1 = p_2 = 0$). It remains to solve the Eq. (4.4) for $3 \leqslant s < t \leqslant R-1$. All coefficients of $\bar{\mathbf{d}}$ in these equations are yet to be determined. Now we can proceed as in the beginning of this proof, i.e. by defining the sets $\mathcal{D}$ and $\mathcal{I}$ in (4.2) and (4.3), respectively, for $3 \leqslant s < t \leqslant R-1$. Note that this amounts to deleting columns $\mathbf{b}_1$ and $\mathbf{b}_2$ from the equations (4.4). For this smaller set of equations, we either arrive at the conclusion that $\mathbf{U}$ has linearly dependent columns or, as above, we are left with a still smaller set of the equations (4.4) to solve. If we do not obtain that $\mathbf{U}$ has linearly dependent columns after a number of such iterations, but keep decreasing the number of equations in (4.4), we will arrive at the situation where $\mathcal{I} = \emptyset$. When this happens, then after each iteration we have deleted two columns of $\mathbf{b}_1, \ldots, \mathbf{b}_{R-1}$ from the equations for which the corresponding elements in $\mathbf{p}$ are zero. Since $k_\mathbf{A} \geqslant 2$ the vector $\mathbf{p}$ should have at least two nonzero elements. Hence, if we end up with $\mathcal{I} = \emptyset$ there are at least the columns $\mathbf{b}_s$, $\mathbf{b}_t$ and $\mathbf{b}_R$ left, for some $1 \leqslant s < t \leqslant R-1$. It follows that $\mathcal{D}$ contains at least (s,t). As we have shown in the beginning of this proof the case $\mathcal{I} = \emptyset$ and $\mathcal{D} \neq \emptyset$ yields a $\mathbf{U}$ with linearly dependent columns. This completes the proof. □

Lemma 4.5 shows that if $R - 1 = r_\mathbf{A} > k_\mathbf{A} \geqslant 2$, then a necessary condition for non-uniqueness is that the condition of Lemma 4.3 does not hold. Next, we show that this is also sufficient for non-uniqueness.

Lemma 4.6. *Let $k_\mathbf{C} = R$, $R - 1 = r_\mathbf{A} > k_\mathbf{A} \geqslant 2$ and $k_\mathbf{B} \geqslant 2$.*

Then we have CP uniqueness if and only if for some s and t, the set $\{\mathbf{b}_s, \mathbf{b}_t, \mathbf{b}_R\}$ is linearly independent and elements p_s and p_t of $\mathbf{p}$ are nonzero.

Proof. We simultaneously reorder the columns of $\mathbf{A}$ and $\mathbf{B}$ such that the first $R-1$ columns of $\mathbf{A}$ are linearly independent. Without loss of generality, we transform $\mathbf{A}$ to $\begin{bmatrix} \mathbf{I}_{R-1} & \mathbf{p} \\ \mathbf{O} & \mathbf{0} \end{bmatrix}$. We assume that the condition of the lemma does not hold. The Eq. (1.5) are $d_s d_t \tilde{\mathbf{b}}_{(s,t)} + d_s d_R p_t \tilde{\mathbf{b}}_{(s,R)} - d_t d_R p_s \tilde{\mathbf{b}}_{(t,R)} = \mathbf{0}$ with $1 \leqslant s < t \leqslant R-1$. We refer to the equation starting with $d_s d_t$ as equation (s,t).

The conditions of the lemma imply $R \geqslant 4$. We have $R - 1 = r_\mathbf{A} > k_\mathbf{A} \geqslant 2$. When $\mathbf{p}$ does not contain any zeros, it follows that $k_\mathbf{A} = r_\mathbf{A}$. When $\mathbf{p}$ contains more than $R-3$ zeros, it is either all-zero (implying $k_\mathbf{A} = 0$) or it is proportional to a column of $\mathbf{I}_{R-1}$ (implying $k_\mathbf{A} = 1$). Hence, the number of zeros in the vector $\mathbf{p}$ is between 1 and $R-3$. We assume without loss of generality that only the first x elements of $\mathbf{p}$ are zero. Since the condition of the lemma does not hold, all sets $\{\mathbf{b}_s, \mathbf{b}_t, \mathbf{b}_R\}$ with $x + 1 \leqslant s < t \leqslant R-1$ must be linearly dependent and, since $k_\mathbf{B} \geqslant 2$, the rank of $\{\mathbf{b}_{x+1}, \ldots, \mathbf{b}_R\}$ equals 2. The proof of Lemma 3.7 then shows that equations (s,t) with $x+1 \leqslant s < t \leqslant R-1$ can be solved for nonzero $d_{x+1}, \ldots, d_R$. Since $p_1 = \cdots = p_x = 0$ the remaining equations are solved for $d_1 = \cdots = d_x = 0$. Hence, condition (1.5) does not hold and we have non-uniqueness. □

4.3. An example

Lemma 4.6 applies to the $3 \times 3 \times 4$ example of Ten Berge and Sidiropoulos [24], that motivated the analysis of Jiang and Sidiropoulos [6]. In this example, $R = 4$, $\mathbf{C} = \mathbf{I}_4$ and

$$\mathbf{A}=\begin{bmatrix}1&0&0&a_1\\0&1&0&a_2\\0&0&1&a_3\end{bmatrix},\quad \mathbf{B}=\begin{bmatrix}1&0&0&b_1\\0&1&0&b_2\\0&0&1&b_3\end{bmatrix}. \tag{4.18}$$

Ten Berge and Sidiropoulos [24] show that if the fourth columns of $\mathbf{A}$ and $\mathbf{B}$ have a zero in different rows, then the CP solution is essentially unique, and if the zeros appear in the same row it is not unique. Using Lemma 4.6, this can be explained as follows. Suppose $a_1=b_2=0$. Then $\{\mathbf{b}_2,\mathbf{b}_3,\mathbf{b}_4\}$ is linearly independent and $a_2a_3\neq 0$. Hence, the condition of Lemma 4.3 holds and we have CP uniqueness. Suppose next that $a_1=b_1=0$. Then any linearly independent set $\{\mathbf{b}_s,\mathbf{b}_t,\mathbf{b}_4\}$ must include $\mathbf{b}_1$. But $a_1=0$ and, hence, the condition of Lemma 4.3 does not hold. CP non-uniqueness now follows from Lemma 4.6. The same example was discussed in Jiang and Sidiropoulos [6] using condition (1.5).

5. Uniqueness in Indscal

We consider the Indscal model as the CP model (1.1)–(1.2) for an $I\times I\times K$ array $\underline{\mathbf{X}}$ with symmetric $I\times I$ slices $\mathbf{X}_k$, and the constraint $\mathbf{A}=\mathbf{B}$ imposed (actually, $\mathbf{A}$ and $\mathbf{B}$ having proportional columns is enough due to the scaling indeterminacy). This constraint is imposed to ensure that the Indscal solution has symmetric slices. However, it is mostly inactive, i.e., the unrestricted CP solution of an array with symmetric slices often has the columns of $\mathbf{A}$ and $\mathbf{B}$ proportional (see Ten Berge and Kiers [23], and Ten Berge et al. [25]). The Indscal model was introduced by Carroll and Chang [1].

Kruskal's condition (1.3) for essential uniqueness in the Indscal case is

$$2k_{\mathbf{A}}+k_{\mathbf{C}}\geqslant 2R+2. \tag{5.1}$$

Note that Kruskal's condition (5.1) also excludes alternative solutions with the columns of $\mathbf{A}$ and $\mathbf{B}$ not being proportional. As for CP, we consider the case where $\mathbf{C}$ has full column rank. Then Kruskal's condition becomes

$$2k_{\mathbf{A}}\geqslant R+2. \tag{5.2}$$

An alternative and more relaxed Indscal uniqueness condition can be obtained using the approach of Jiang and Sidiropoulos [6]. Namely, if $k_{\mathbf{C}}=R$, then the Indscal solution $(\mathbf{A},\mathbf{C})$ is essentially unique if and only if

$$(\mathbf{A}\odot\mathbf{A})\mathbf{d} \text{ is not of the form } (\mathbf{f}\otimes\mathbf{f}) \text{ for any } \mathbf{d} \text{ with } \omega(\mathbf{d})\geqslant 2, \tag{5.3}$$

where $\mathbf{d}=(d_1,\ldots,d_R)^T$. Note that (5.3) is analogous to (1.4). As for CP, condition (5.3) is equivalent to (1.5) with $\mathbf{U}=\widetilde{\mathbf{A}}\odot\widetilde{\mathbf{A}}$. However, in the Indscal case the matrix $\mathbf{U}$ contains more redundant rows than for CP, as was noted by Stegeman et al. [21]. After these redundant rows have been deleted from $\mathbf{U}$, the matrix is vertical or square if and only if

$$\frac{I(I-1)}{4}\left(\frac{I(I-1)}{2}+1\right)-\binom{I}{4}\geqslant\frac{R(R-1)}{2}, \tag{5.4}$$

where the term $\binom{I}{4}$ only appears if $I\geqslant 4$. Stegeman et al. [21] give a partial proof of the condition (5.4) being sufficient for $\mathbf{U}$ having full column rank almost surely when $\mathbf{A}$ is randomly sampled from an IR-dimensional continuous distribution.

In Section 5.1 below, we present the Indscal versions of the CP uniqueness results from Sections 3 and 4. Section 5.2 contains a necessary and sufficient Indscal uniqueness condition for the case $k_{\mathbf{C}}=R$, $r_{\mathbf{A}}=R-1$ and $k_{\mathbf{A}}\geqslant 2$.

5.1. Kruskal's condition and U *for Indscal*

The CP uniqueness results from Sections 3 and 4 can be translated to the Indscal case by setting $\mathbf{A}=\mathbf{B}$. Our main result is that Kruskal's condition (5.2) implies that $\mathbf{U}=\widetilde{\mathbf{A}}\odot\widetilde{\mathbf{A}}$ has full column rank.

Theorem 5.1. *Let $k_{\mathbf{C}}=R$ and $k_{\mathbf{A}}\geqslant 2$.*

If $2k_{\mathbf{A}}\geqslant R+2$, then $\mathbf{U}=\widetilde{\mathbf{A}}\odot\widetilde{\mathbf{A}}$ has full column rank and, hence, we have Indscal uniqueness. $\square$

However, we are able to obtain the following stronger Indscal uniqueness result.

Proposition 5.2. *Let* $k_{\mathbf{C}} = R$ *and* $k_{\mathbf{A}} \geqslant 2$.
If $r_{\mathbf{A}} + k_{\mathbf{A}} \geqslant R + 2$, *then* $\mathbf{U} = \widetilde{\mathbf{A}} \odot \widetilde{\mathbf{A}}$ *has full column rank and, hence, we have Indscal uniqueness.*

Proof. If $r_{\mathbf{A}} = k_{\mathbf{A}}$, the proof follows from Proposition 3.2 with $\mathbf{A} = \mathbf{B}$. If $r_{\mathbf{A}} > k_{\mathbf{A}}$, the proof follows from Proposition 4.2 with $\mathbf{A} = \mathbf{B}$. $\square$

Since the k-rank of a matrix cannot be larger than its rank, Proposition 5.2 implies Theorem 5.1. If $r_{\mathbf{A}} > k_{\mathbf{A}}$, then the uniqueness condition of Proposition 5.2 is more relaxed than Kruskal's condition (5.2). However, in this case an even weaker uniqueness condition is provided by the analogue of Lemma 4.3 presented as Lemma 5.3 below. We assume the columns of $\mathbf{A}$ are ordered such that the first $r_{\mathbf{A}}$ columns are linearly independent. Without loss of generality we transform $\mathbf{A}$ to $\begin{bmatrix} \mathbf{I}_{r_{\mathbf{A}}} & \mathbf{P} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$. Let $\mathbf{e}_k$ denote a column of $\mathbf{I}_{r_{\mathbf{A}}}$.

Lemma 5.3. *Let* $k_{\mathbf{C}} = R$ *and* $r_{\mathbf{A}} > k_{\mathbf{A}} \geqslant 2$. *If the set* $\{\mathbf{e}_s, \mathbf{e}_t, \mathbf{P}\}$ *is linearly independent for some s and t, and rows s and t of* $\mathbf{P}$ *and row* (s,t) *of* $\widetilde{\mathbf{P}}$ *contain no zeros, then* $\mathbf{U}$ *has full column rank and, hence, we have Indscal uniqueness.* $\square$

An example $\mathbf{A}$ for which the condition in Lemma 5.3 holds but not the condition in Proposition 5.2 is

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 0 & 4 \end{bmatrix}. \tag{5.5}$$

Indeed, we have $r_{\mathbf{A}} = 4$, $k_{\mathbf{A}} = 3$ and $R + 2 = 8$. But the set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{P}\}$ is linearly independent, the first two rows of $\mathbf{P}$ contain no zeros and row $(1,2)$ of $\widetilde{\mathbf{P}}$ equals 1.

5.2. Necessary and sufficient Indscal uniqueness conditions

Here, we give some necessary and sufficient Indscal uniqueness conditions. First, we consider the case where $\mathbf{A}$ has full column rank.

Lemma 5.4. *Let* $k_{\mathbf{A}} = R$. *Then we have Indscal uniqueness if and only if* $k_{\mathbf{C}} \geqslant 2$.

Proof. Analogous to (1.4) we have Indscal uniqueness if and only if $(\mathbf{A} \odot \mathbf{C})\mathbf{d}$ is not of the form $(\mathbf{f} \otimes \mathbf{g})$ for any $\mathbf{d}$ with $\omega(\mathbf{d}) \geqslant 2$. This is equivalent to (1.5) with $\mathbf{U} = \widetilde{\mathbf{A}} \odot \widetilde{\mathbf{C}}$. As we showed below (3.1), this holds if $k_{\mathbf{C}} \geqslant 2$. The proof is complete by observing that $k_{\mathbf{C}} \geqslant 2$ is necessary for CP or Indscal uniqueness (see e.g. Stegeman and Sidiropoulos [22]). $\square$

Note that the condition of Lemma 5.4 is identical to Kruskal's condition (5.1) in this case. Next, we consider the case where $\mathbf{C}$ has full column rank. We have the following analogue of Lemmas 3.7 and 4.6.

Lemma 5.5. *Let* $k_{\mathbf{C}} = R$, $r_{\mathbf{A}} = R - 1$ *and* $k_{\mathbf{A}} \geqslant 2$.
Then we have Indscal uniqueness if and only if $k_{\mathbf{A}} \geqslant 3$.

Proof. Since $r_{\mathbf{A}} + k_{\mathbf{A}} \geqslant R - 1 + 3 = R + 2$, sufficiency follows from Proposition 5.2. Next, we show that $k_{\mathbf{A}} = 2$ implies non-uniqueness. We permute the columns of $\mathbf{A}$ such that the first $R - 1$ columns are linearly independent. Without loss of generality we transform $\mathbf{A}$ to $\begin{bmatrix} \mathbf{I}_{R-1} & \mathbf{p} \\ \mathbf{O} & \mathbf{0} \end{bmatrix}$. For $\mathbf{U} = \widetilde{\mathbf{A}} \odot \widetilde{\mathbf{A}}$, Eq. (1.5) is

$$d_s d_t + d_s d_R p_t^2 + d_t d_R p_s^2 = 0, \quad 1 \leqslant s < t \leqslant R-1, \tag{5.6}$$
$$d_s d_R p_g p_h = 0, \quad g, h \neq s,\ g \neq h,\ 1 \leqslant s \leqslant R-1. \tag{5.7}$$

Since $k_{\mathbf{A}} = 2$ the vector $\mathbf{p}$ contains only two nonzero elements, say p_1 and p_2. It can be verified that (5.6)–(5.7) are satisfied if $d_3 = \cdots = d_{R-1} = 0$ and

$$d_1 d_2 + d_1 d_R p_2^2 + d_2 d_R p_1^2 = 0. \tag{5.8}$$

Since there always exist nonzero d_1, d_2 and d_R such that equation (5.8) holds, condition (1.5) does not hold and, hence, we have non-uniqueness. □

Lemma 5.5 shows that the condition of Proposition 5.2 is also necessary for Indscal uniqueness if $r_{\mathbf{A}} = R-1$. This also shows that the conditions of Proposition 5.2 and Lemma 5.3 are equivalent for $r_{\mathbf{A}} = R-1$.

When $\mathbf{A}$ is transformed to a matrix with many zeros, the matrix $\mathbf{U} = \widetilde{\mathbf{A}} \odot \widetilde{\mathbf{A}}$ also has many zeros. This makes it easier to analyze Indscal uniqueness than CP uniqueness.

6. Discussion

In this paper, we studied the essential uniqueness of CP and Indscal decompositions when component matrix $\mathbf{C}$ has full column rank. We analyzed the relation between Kruskal's [11] uniqueness condition and the recent uniqueness conditions of Jiang and Sidiropoulos [6] and De Lathauwer [2]. It is known that for random $(\mathbf{A}, \mathbf{B})$ the almost sure uniqueness condition of De Lathauwer [2] is more relaxed than Kruskal's condition. And it is generally believed that this is also true for non-random $(\mathbf{A}, \mathbf{B})$, although a proof has not been provided. The main contribution of our paper is that we provided this proof. In particular, we showed that Kruskal's condition implies that the matrix $\mathbf{U}$ of Jiang and Sidiropoulos [6] has full column rank, which makes the latter a weaker sufficient uniqueness condition. Moreover, we obtained new Kruskal-type uniqueness conditions for CP and Indscal (Propositions 3.2, 4.2 and 5.2) that are weaker than Kruskal's condition but stronger than $\mathbf{U}$ having full column rank.

Also, we obtained weaker uniqueness conditions (Lemmas 4.3 and 5.3) by allowing (for CP) more subtle interactions between the component matrices $\mathbf{A}$ and $\mathbf{B}$. For the case $r_{\mathbf{A}} = R-1$, we proved several necessary and sufficient uniqueness conditions for CP and Indscal (Lemmas 3.7, 4.6 and 5.5). These are equivalent to $\mathbf{U}$ having full column rank. The results are not difficult to extend to the case $r_{\mathbf{A}} = R-2$. However, this requires tedious analysis of all possible dependencies in the columns of $\mathbf{A}$ and $\mathbf{B}$. We expect that for $r_{\mathbf{A}} \leqslant R-2$ necessary and sufficient CP uniqueness conditions can be obtained that are weaker than $\mathbf{U}$ having full column rank.

In our analysis, we have confined ourselves to the real-valued CP and Indscal decompositions. However, our proofs can easily be adapted to the complex-valued CP and Indscal decompositions. A proof of Kruskal's uniqueness condition for complex-valued CP can be found in Sidiropoulos and Bro [13]. The necessary and sufficient uniqueness condition of Jiang and Sidiropoulos [6] and the condition of $\mathbf{U}$ having full column rank are also valid for the complex-valued CP decomposition.

Stegeman and Ten Berge [20] give two examples of unique CP solutions ($3 \times 3 \times 5$ with $R = 5$ and $3 \times 4 \times 6$ with $R = 6$) with the ranks and k-ranks of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ being equal, that do not satisfy Kruskal's uniqueness condition. Since $\mathbf{C}$ has full column rank, these examples are within the scope of our analysis. However, in both cases the CP uniqueness condition of Proposition 3.2 does not hold. In the first example the matrix $\mathbf{U}$ is 9×10 and it cannot have full column rank. In the second example $\mathbf{U}$ is 18×15 and it follows from De Lathauwer [2] that $\mathbf{U}$ has full column rank almost surely when $\mathbf{A}$ and $\mathbf{B}$ are randomly sampled from an $(I+J)R$-dimensional continuous distribution. To include these examples in our analysis requires an extension of Lemma 3.7 to the case $r_{\mathbf{A}} = R-2$.

Contrary to CP, we have not found unique Indscal solutions for which the matrix $\mathbf{U} = \widetilde{\mathbf{A}} \odot \widetilde{\mathbf{A}}$ has linearly dependent columns. This raises the question whether such Indscal solutions exist, i.e., whether

U having full column rank is also necessary for Indscal uniqueness. Further research is needed to answer this question.

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